THE

MATHEMATICAL GAZETTE.

EDITED BY

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WITH THE CO-OPERATION OF

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TWO ILLUSTRATIONS OF ELIMINATION.

1. If A, B, C, G are four given points in a plane, it is possible to place masses x, y, z at A, B, C respectively, so that the centroid of the system is at G. This, it is to be observed, can only be done in one way as x, y, z must be proportional to the areal (or "barycentric") co-ordinates of G with respect to the triangle ABC.

If, then, P is any variable point in the same plane,

$$\Sigma x PA^2 - PG^2 \Sigma x$$

is independent of P (the "moments of inertia" theorem due to Euler).

2. Applying this theorem to each of the points A, B, C, and G (whose distances from A, B, C are p, q, r respectively)

$$\begin{aligned}
&+c^2y+b^2z-(x+y+z)p^2=K,\\
c^2x&+a^2z-(x+y+z)q^2=K,\\
b^2x+a^2y&-(x+y+z)r^2=K,\\
p^2x+q^2y+r^2z&=K,\\
\text{also}\quad x+y+z&-(x+y+z)=0.
\end{aligned}$$

3. Eliminating $x, y, z, -(x+y+z)$, and $-K$ from these equations:

$$\begin{vmatrix}
0, & c^2, & b^2, & p^2, & 1\\
c^2, & 0, & a^2, & q^2, & 1\\
b^2, & a^2, & 0, & r^2, & 1\\
p^2, & q^2, & r^2, & 0, & 1\\
1, & 1, & 1, & 1, & 0
\end{vmatrix}=0,$$

or $a^2p^4+b^2q^4+c^2r^4=(b^2+c^2-a^2)(q^2r^2+a^2p^2)+\ \ldots\ +\ \ldots\ -a^2b^2c^2$,

the relation connecting p, q, r which can be otherwise obtained as in Todhunter's *Spherical Trigonometry*, and Levett and Davison's *Trig.*, p. 172.

4. In the *Educational Times*, Reprint Vol. LXVII., *Q.* 13424, p. 113, I have shewn that the moments of inertia theorem takes, in the case of three points A, B, C only, a simpler form

$$\Sigma x \,.\, PA^2 = (PG^2 - GT^2)\Sigma x,$$

where GT is the tangent from G to the circum-circle.

If, therefore, A, B, C, G be concyclic, $K=0$; and the relation becomes

$$\begin{vmatrix} 0, & c^2, & b^2, & p^2 \\ c^2, & 0, & a^2, & q^2 \\ b^2, & a^2, & 0, & r^2 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

or

$$ap \pm bq \pm cr = 0.$$

5. Now let A, B, C, D be four points in a plane at which masses x, y, z, u are respectively placed; G the centroid of the system.

From A, B, C, D as centres describe circles of radii a, b, c, d and from G as centre describe a circle of radius

$$\Sigma xa / \Sigma x (\equiv g).$$

Then if P be the centre and r the radius of any variable circle,

$$\Sigma x(PA)^2 - (PG)^2 \Sigma x$$

is independent of both P and r; (PA) being the length of the common tangent to the circles (P, r) and (A, a).

For the above expression

$$= \Sigma x \{PA^2 - (r-a)^2\} - \{PG^2 - (r-g)^2\} \,.\, \Sigma x$$

$$= \Sigma x PA^2 - PG^2 \,.\, \Sigma x + \text{terms in } r^2 \text{ and } r \text{ (which cancel out)} + \text{constant}$$

$$= \text{constant}.$$

This relation is of the same form as that in (1).

6. Applying in a similar manner the theorem to A, B, C, D, and G, we obtain the relation between the common tangents to five circles as given in Salmon:

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1, & 1 \\ 1, & 0, & (12)^2, & (13)^2, & (14)^2, & (15)^2 \\ 1, & (12)^2, & 0, & (23)^2, & (34)^2, & (25)^2 \\ 1, & (13)^2, & (23)^2, & 0, & (34)^2, & (35)^2 \\ 1, & (14)^2, & (24)^2, & (34)^2, & 0, & (45)^2 \\ 1, & (15)^2, & (25)^2, & (35)^2, & (45)^2, & 0 \end{vmatrix} = 0.$$

7. It is to be observed that A, B, C, D, G being centres of given circles of radii, a, b, c, d, g respectively, we can arrange the masses x, y, z, u to be placed at A, B, C, D so that G shall be the centroid of the system in an infinite number of ways.

We can, therefore, so arrange them that not only is G the centroid but also $\Sigma xa = g\Sigma x$ as well. The foregoing proof is therefore perfectly general.

R. F. DAVIS.

NOTE ON THE SOLUTION OF CUBIC AND BIQUADRATIC EQUATIONS.

1. Let $$x^3+px^2+qx+r=0$$ be a cubic equation, whose roots are α, β, γ. Then, defining the quantity λ by means of the equation $\lambda^3=1$, we have

$$\begin{aligned}(\alpha+\lambda\beta+\lambda^2\gamma)^3 &= \alpha^3+\beta^3+\gamma^3+6\alpha\beta\gamma+3\lambda(\beta^2\gamma+\gamma^2\alpha+\alpha^2\beta)\\ &\quad +3\lambda^2(\beta\gamma^2+\gamma\alpha^2+\alpha\beta^2)\\ &= A+\lambda B+\lambda^2 C.\end{aligned}$$

Since this is true for all values of λ that satisfy the equation $\lambda^3=1$; we have, ω being one of the imaginary cube roots of unity, the three equations

$$\alpha+\beta+\gamma=(A+B+C)^{\frac{1}{3}},$$

$$\alpha+\omega\beta+\omega^2\gamma=(A+\omega B+\omega^2 C)^{\frac{1}{3}},$$

$$\alpha+\omega^2\beta+\omega\gamma=(A+\omega^2 B+\omega C)^{\frac{1}{3}}.$$

From these equations it follows that

$$3\alpha=(A+B+C)^{\frac{1}{3}}+(A+\omega B+\omega^2 C)^{\frac{1}{3}}+(A+\omega^2 B+\omega C)^{\frac{1}{3}},$$

$$3\beta=(A+B+C)^{\frac{1}{3}}+\omega^2(A+\omega B+\omega^2 C)^{\frac{1}{3}}+\omega(A+\omega^2 B+\omega C)^{\frac{1}{3}},$$

$$3\gamma=(A+B+C)^{\frac{1}{3}}+\omega(A+\omega B+\omega^2 C)^{\frac{1}{3}}+\omega^2(A+\omega^2 B+\omega C)^{\frac{1}{3}},$$

Thus, if A, B, C can be expressed in terms of the coefficients of the given cubic equation, we have a solution of the said equation. Now A is a symmetrical function of the roots of the said equation, and may therefore be so expressed. In fact we have

$$A=-p^3+3pq-qr.$$

Writing $$U=\beta^2\gamma+\gamma^2\alpha+\alpha^2\beta,$$

and $$V=\beta\gamma^2+\gamma\alpha^2+\alpha\beta^2,$$

we see that $U+V$ is a symmetrical function of the roots. In fact we have

$$U+V=-pq+3r. \qquad \text{.....(1)}$$

Also $$\begin{aligned}U-V &= \beta\gamma(\beta-\gamma)+\gamma\alpha(\gamma-\alpha)+\alpha\beta(\alpha-\beta)\\ &= -(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta).\end{aligned}$$

This shows that $(U-V)^2$ is a symmetrical function of the roots; and, as $U+V$ is of the same character, it follows that UV must be so also. In fact we have

$$\begin{aligned}UV &= \beta^3\gamma^3+\gamma^3\alpha^3+\alpha^3\beta^3+3\alpha^2\beta^2\gamma^2+\alpha\beta\gamma(\alpha^3+\beta^3+\gamma^3)\\ &= p^3r+q^3-6pqr+qr^2. \qquad \text{.....(2)}\end{aligned}$$

From equations (1) and (2) the values of U and V can be readily obtained by a known method. Thus the values of B and C, in terms of the coefficients of the equation, can be found, and the solution completed.

2. Next suppose that we have a biquadratic equation

$$x^4+px^3+qx^2+rx+s=0,$$

whose roots are α, β, γ, δ. This time we will employ two quantities, λ and μ, defined by the equations

$$\lambda^2=1, \quad \mu^2=1.$$

We have $$(\alpha+\lambda\beta+\mu\gamma+\lambda\mu\delta)^2=\alpha^2+\beta^2+\gamma^2+\delta^2+2\lambda(\alpha\beta+\gamma\delta)$$
$$+2\mu(\alpha\gamma+\beta\delta)+2\lambda\mu(\alpha\delta+\beta\gamma)$$
$$=A+\lambda B+\mu C+\lambda\mu D.$$

Now this equation is satisfied by any pair of possible values of λ and μ. We have, therefore, four cases, viz.,

$$\lambda=1, \quad \mu=1,$$
$$\lambda=-1, \quad \mu=1,$$
$$\lambda=1, \quad \mu=-1,$$
$$\lambda=-1, \quad \mu=-1.$$

Thus we obtain
$$\alpha+\beta+\gamma+\delta=(A+B+C+D)^{\frac{1}{2}},$$
$$\alpha-\beta+\gamma-\delta=(A-B+C-D)^{\frac{1}{2}},$$
$$\alpha+\beta-\gamma-\delta=(A+B-C-D)^{\frac{1}{2}},$$
$$\alpha-\beta-\gamma+\delta=(A-B-C+D)^{\frac{1}{2}}.$$

Thus, if A, B, C, D can be expressed in terms of the coefficients of the equation, we readily obtain the solution in the following form :

$$4\alpha=(A+B+C+D)^{\frac{1}{2}}+(A-B+C-D)^{\frac{1}{2}}+(A+B-C-D)^{\frac{1}{2}},$$
$$+(A-B-C+D)^{\frac{1}{2}},$$
$$4\beta=(A+B+C+D)^{\frac{1}{2}}-(A-B+C-D)^{\frac{1}{2}}+(A+B-C-D)^{\frac{1}{2}},$$
$$-(A-B-C+D)^{\frac{1}{2}},$$
$$4\gamma=(A+B+C+D)^{\frac{1}{2}}+(A-B+C-D)^{\frac{1}{2}}-(A+B-C-D)^{\frac{1}{2}},$$
$$-(A-B-C+D)^{\frac{1}{2}},$$
$$4\delta=(A+B+C+D)^{\frac{1}{2}}-(A-B+C-D)^{\frac{1}{2}}-(A+B-C-D)^{\frac{1}{2}},$$
$$+(A-B-C+D)^{\frac{1}{2}}.$$

Now A is a symmetrical function of the roots, and we have
$$A=p^2-2q.$$

The solution of the equation, therefore, depends upon the expression of the three quantities
$$\alpha\beta+\gamma\delta, \quad \alpha\gamma+\beta\delta, \quad \alpha\delta+\beta\gamma$$
in terms of the coefficients of the equation. Now these three quantities clearly form a group, *i.e.* they are such that any interchanges effected among the quantities α, β, γ, δ leave us with the same three quantities. Thus it is clear that if we form a cubic equation with these three quantities for roots, its coefficients will be expressible in terms of the coefficients of the given equation. In fact, writing U, V, W for the three quantities, we have

$$U+V+W=\Sigma\alpha\beta=q,$$
$$VW+WU+UV=\Sigma\alpha^2\beta\gamma=pr-4s,$$
$$UVW=\Sigma\alpha^3\beta\gamma\delta+\Sigma\alpha^2\beta^2\gamma^2$$
$$=p^2s-4qs+r^2.$$

Thus the auxiliary cubic, whose roots are U, V, W, is
$$x^3-qx^2+(pr-4s)x-p^2s+4qs-r^2=0.$$

The solution of this equation will give U, V, W, and therefore B, C, D in terms of the coefficients of the biquadratic. J. BRILL.

REVIEWS.

A Treatise on Elementary Dynamics, dealing with relative motion mainly in two dimensions, by H. A. ROBERTS, M.A. Macmillan & Co., 1900, pp. xi. + 258.

The Student's Dynamics, comprising Statics and Kinetics, by G. M. MINCHIN, M.A., F.R.S. George Bell & Sons, 1900, pp. xii. + 258.

Elementary Mechanics of Solids, by W. T. A. EMTAGE, M.A. Macmillan & Co., 1900, pp. viii. + 333.

Mr. Roberts's book is the most ambitious of these three. It does not profess to belong to the most elementary class of text-books on the subject; and the author, while restricting himself to the scope indicated by the title, makes a serious attempt to make good the ground he occupies, and to tackle the subject with a sense of responsibility towards students of science. A notable feature of the book is the way in which, in the discussion of dynamical principles, and the application of them to practical questions, some care is taken to specify the base relative to which motion is reckoned. Thus an old-established source of perplexity to students of the subject is to a considerable extent avoided. The author adopts the term "Galileo's axes" as the name for a base attached to the surface of the earth; and, by the use of this term, is able to call attention, in a convenient way, to the fact that the ordinary theory applied, without correction, to motion relative to the earth is only approximate, without introducing any complicated calculations on the subject at too early a stage. An extension to the treatment of motion relative to other bases, such as a base suitable for the whole solar system, which Mr. Roberts calls "Newton's axes," is easily and naturally made. This way of developing the subject seems to be satisfactory, and Mr. Roberts's treatment is on the whole decidedly good, though no doubt some trifling improvements could be suggested. The establishment of mass to express acceleration-ratios is well done.

Another important feature is a certain amount of use of vector notation, which is quite suitable to a book of this class, and helps it a great deal. It is a curious thing how slow writers of elementary text-books have been to take advantage of the attractive features presented by such treatment, which, kept within due limits, is certain to interest an intelligent boy. We notice that Mr. Roberts is apt to refer to the motion of a point relative to another *point*, where motion relative to a certain base or axes is what is really meant. This form of words is suitable, for the sake of brevity, on some occasions; but we think it occurs too often, and is the cause of some obscurity. It is to be found also in Professor Minchin's book. Mr. Roberts makes some use of the hodograph for transforming a theorem about radius and velocity into one about velocity and acceleration, and calls attention to the scope of this convenient method. The book contains a large number of useful examples.

Professor Minchin's book is intentionally more elementary. Some parts of it are not without merit. There is a good deal of clear

explanation, helped by arithmetical examples. But in some respects it is not a satisfactory book. Some neeessary explanations are altogether omitted. We find no explanation of acceleration except for the case of rectilinear motion; and yet the direction of acceleration, and finally normal acceleration are dealt with. The treatment of the fundamental principles of dynamics is incomplete and not satisfactory. And throughout the book occasional inaccurate statements are to be found; such as, on p. 94, that a force can do negative work only if some other force is acting on the body, and, on p. 244, that Hooke's law holds for a steel bar with extension 1/100.

Anyone would suppose from the preface that Professor Minchin had adopted the course of defining force in terms of acceleration. But, after reading the book, it is not easy to make out what the procedure is which the preface refers to. As a matter of fact, Professor Minchin, though he refers to acceleration for the specification of a unit, a dyne—employs the method of defining the measurement of force by the superposition of conditions, and treats its relation to acceleration as an experimental result. He refers to spiral springs in this connection to give definiteness to the explanations and diagrams. We think that this course has advantages within a certain limited range of treatment, and we do not altogether condemn it; but the way in which Professor Minchin connects forces, indicated by springs, with acceleration and masses is incomplete and not clear. There is no specification of the base relative to which the accelerations which he refers to are to be measured, nor is the question raised as to how far the specification of a base is needed. Weight is adopted as the test of equality of masses of different substances, a retrograde procedure, and one applicable only to terrestrial bodies. On p. 16 we find it tacitly assumed that equal masses, tested in this way, of all substances will give the same dyne. Then we are told to assume that, for a given mass, acceleration is proportional to force measured by superposition of springs. In fact there are not sufficient indications of the character of the steps which lead to the result which is aimed at, $P = m\alpha$.

Professor Minchin says, in his preface, that he has "made another effort to banish that extremely misleading term *centrifugal force*"; so we naturally look to see what he has done to accomplish this. On p. 211 he says, "as regards the tendency of a revolving body to fly outwards from the centre, no such thing exists." He can hardly expect any reader who has ever seen the governor of a steam-engine, or the behaviour of a system attached to a whirling table, to agree with this statement. Surely this is not the way to approach the subject. In the whole of his discussion Professor Minchin does not even mention the two points of real importance, namely: (1) that centrifugal force is exhibited only in cases of rotation relative to the directions of the fixed stars, and (2) that it is condemned as a force by the law of action and reaction. Indeed, we actually find, on the same page, 211, the statement that, in the case of a heavy body on the earth, the earth experiences a pull which is "the exact equal and opposite" of the weight of the body. Now, the earth is an example of a whirling table, and the component of weight, which is due to the rotation, is a good example of the cases in which centri-

fugal force (by whatever name it is called) is useful. Professor Minchin's treatment, in fact, seems to us to be calculated to land his readers in hopeless perplexity.

On p. 18 Professor Minchin has an article on the nature of weight. He discusses the law of gravity as applied to a sphere, and how gravitation varies at different distances from the centre of the earth, and tells us that, in connection with a body on the surface of the earth, it is called the weight of the body. He goes on to say that weight, thus defined, varies at different places on the earth because the earth is not quite spherical. He then tells us how this variation of weight might be measured, and that the greatest amount of variation is $\frac{1}{2}$ per cent. A reader of this article might be surprised to learn that, as a matter of fact, though the total variation of weight between the equator and the pole is about $\frac{1}{2}$ per cent., only about one-third of this variation is due to the law of gravitation and the flattening of the poles, the remaining two-thirds being due to centrifugal force. Mr. Roberts deals to some extent with the same topic, but the point which he discusses in detail is the centrifugal force part of the variation. He might very well have mentioned the part due to the law of gravitation, and it was rather a pity to omit a reference to this. But his statements on the subject are clear enough; a calculation is made on the assumption that the earth is spherical, and he mentions both the whole variation of weight and the part due to centrifugal force. We do not, however, think he is right in suggesting, as he does in one of the examples on p. 122, that a railway train travelling west is heavier than when travelling east; this appears to be introducing into weight an element which does not belong to it. In Mr. Emtage's book we find both parts of the variation of gravity explained in general terms.

A peculiarity of Professor Minchin's book is that he adopts (see p. 56) the plan of defining his symbols for lengths, forces, etc., as representing concrete lengths, forces, etc., and not merely the numerical values of them. Accordingly he explains the necessary homogeneity of equations in symbols of one kind, not as arising out of certain liberty which we have reserved for ourselves as to choice of units, which is the usual point of view, but as arising out of the impossibility of two concrete things of different kinds being equal. He says that the cube of a length is a volume, but does not attempt to construct a complete calculus for symbols treated in this way. This is starting on an impracticable enterprise, and it is hardly necessary to say that it is not carried through consistently. Every case in which a symbol for force or velocity is equated to a mere number is a violation of the system, and there are many such cases throughout the book. The system appears to us to be a very bad one, introducing, as it does, quite unnecessarily, difficulties of interpretation of combinations of symbols.

Mr. Emtage's book, we are told in the preface, follows a South Kensington syllabus, but contains some additional matter. A notable feature of it is the experimental illustrations, which are carefully described, with a view to their being performed with very simple apparatus. Another peculiarity of it is the way in which the subject

is developed by the aid of independent experimental, so-called, proofs of the parallelogram of forces, equation of moments, and so on, making a number of different points of departure. In some cases we are given theoretical proofs afterwards, but we cannot find even a mention of the possibility of any theoretical investigation of the parallelogram of forces. We gather that this peculiarity is derived from the South Kensington syllabus. The book seems to be a good one for the purpose for which it is intended, and likely to be useful.

W. H. MACAULAY.

[V.] ON MARKING EUCLID PAPERS.

THE following is a summary of the answers received on this question—some from correspondents of considerable experience as examiners.

Question.	Largest Deduction.	Least Deduction.	Average of all Replies.
1. a.	20	5	10
1. b.	90	5	17·2
2. a.	75	10	24·5
2. b.	10	0	1 nearly.

The very considerable differences of opinion indicated above, probably arise from the fact that some examiners, starting from a perfect answer, deduct marks for every defect in order to do justice to the perfect answer, while others, starting from the average pupil, desire to give a few marks for anything approaching to a correct answer.

Again, it is sometimes doubtful whether a particular answer discloses a mere slip or a grave mistake in principle. Some examiners give the benefit of the doubt to the answer, others to those who have given the correct answer. *On the whole* the replies seem to indicate that the most experienced examiners take the most lenient view of mistakes.

Some correspondents who desire the overthrow of Euclid see an argument in favour of their view. The fact is, that in marking or teaching Euclid we are really dealing with Geometry *and Logic*. Hence some who denounce Euclid, say they could teach or examine Geometry better without him, which, no doubt, in a sense is true. Also others who, seeing no hope for the inclusion of Logic apart from Euclid in our scheme of education, and regarding Logic, the art of reasoning correctly on unfamiliar matters, as a very important part of education, cling to Euclid, with all the concomitant disadvantages.

The question appears to show very clearly the necessity in important competitive examinations of having the scale of marking settled by two or more examiners in conjunction, and the great assistance which would be furnished by a *vivâ voce* in deciding whether a given reply involves a slip or a mistake. Competitive mathematical examinations largely govern the mathematical education of the country, and in some respects (notably in fostering the craze for short methods and dodges) have a very bad effect. Every examiner and teacher knows how very large the probable error of the marks is in any examination conducted by *individuals* (as are all Civil Service Commission examinations), but this is not so well known as it ought to be to the general public. The assumed infallibility of the examination test has led to the gradual rejection of all other modes of selection, and to a general acquiescence in the effects produced by examinations.

C. S. J.

MATHEMATICAL NOTES.

92. [K. 13. a.] *Solution of Problem* 372, *and notes suggested thereby.*

No solution of this having yet appeared, I add the following analytical notes to my previous statement.

The general (finite) displacement of a rigid body by a screw about the axis of z is given by

$$x'=\alpha x, \quad y'=\frac{1}{\alpha}y, \quad z'=z+\beta,$$

where

$$x=X+iY, \quad y=X-iY;$$

X, Y, z are a rectangular Cartesian set of coordinates and α, β are constants, of which β is real and the absolute value of α is unity; in fact $\alpha=e^{i\phi}$, if ϕ be the angle of rotation. Then a point midway between $(x'y'z')$ and (xyz) is

$$x_0=\tfrac{1}{2}(1+\alpha)x, \quad y_0=\frac{1}{2\alpha}(1+\alpha)y, \quad z_0=z+\tfrac{1}{2}\beta,$$

so that if (xyz) is in a given plane

$$Ax+By+Cz+D=0,$$

then $(x_0y_0z_0)$ is in another plane.

The plane bisecting at right angles the line joining (xyz), $(x'y'z')$ is

$$(x'-x)(\eta-y_0)+(y'-y)(\xi-x_0)+2(z'-z)(\zeta-z_0)=0,$$

(ξ, η, ζ) being current coordinates in the plane. This reduces to

$$(1-\alpha)\left(\frac{1}{\alpha}\xi y-x\eta\right)+\beta(2\zeta-2z-\beta)=0.$$

Hence with (xyz) in the given plane, the latter plane contains the fixed point (O)

$$\frac{-(1-\alpha)\eta}{A}=\frac{(1-\alpha)\xi}{\alpha B}=\frac{-2\beta}{C}=\frac{\beta(2\zeta-\beta)}{D},$$

which completes the solution of the problem.

If O has been displaced from the point O', we have

$$\xi'=\frac{1}{\alpha}\xi=-\frac{2\beta}{1-\alpha}\frac{B}{C}, \quad \eta'=\alpha\eta=\frac{2\beta\alpha}{1-\alpha}\frac{A}{C}, \quad \zeta'=\zeta-\beta=-\frac{\beta}{2}-\frac{D}{C},$$

and with these values it can be proved (just as above) that the plane

$$Ax+By+Cz+D=0$$

bisects OO' at right angles; this is the property of O stated in my last note (*Gazette*, Oct., 1900).

If we desire to evaluate the *velocities* due to an instantaneous screw about the axis of z, we have to make β, ϕ infinitesimal, and then we find

$$\frac{dx}{dt}=i\theta x, \quad \frac{dy}{dt}=-i\theta y, \quad \frac{dz}{dt}=\gamma,$$

where $\theta=\dfrac{d\phi}{dt}$, $\gamma=\dfrac{d\beta}{dt}$, so that both are real; and the pitch of the screw is γ/θ.

In these coordinates we can prove very easily a theorem due to Cayley, that every point $(x_0y_0z_0)$ can move instantaneously along the line joining $(x'y'z')$, (xyz) for all positions of (xyz). For, in terms of $(x_0y_0z_0)$,

$$x'-x=2\frac{\alpha-1}{\alpha+1}x_0, \quad y'-y=-2\frac{\alpha-1}{\alpha+1}y_0, \quad z'-z=\beta,$$

and thus the point $(x_0y_0z_0)$ can be screwed along this line by an infinitesimal screw about the axis of z of pitch $\frac{\beta i}{2}\frac{\alpha+1}{\alpha-1}=\frac{\beta}{2}\cot\frac{\phi}{2}$ (using the expressions for the velocities just found). So that, in particular, if the original motion be infinitesimal, this value of the pitch is (β/ϕ) as it ought to be.

In concluding, it may be well to point out wherein the three-dimensional results differ from the two-dimensional. In the two-dimensional case, we have only to ignore z and β in the above; so that the line bisecting at right angles the join of (xy), $(x'y')$, or

$$(x'-x)(\eta-y_0)+(y'-y)(\xi-x_0)=0,$$

reduces to

$$\frac{1}{a}\xi y-x\eta=0,$$

which passes through the origin whatever the point (x, y) may be; thus the origin is a marked point of the body which does not move in the displacement. This is of course a familiar property, and is only mentioned now for comparison with the other results.

If we assume that the usual conditions for orthogonality of lines, etc., can be extended to space of n dimensions, merely by adding on terms of the same type; I have proved that when n is *odd* there is no real point at a finite distance which does not move in a general displacement of a rigid body; but if n is *even*, there is always at least one such point (cf. R. F. Muirhead, *Proc. Edin. Math. Soc.*, Vol. XVI., p. 70). Of course by specializing the motion we can always keep a certain number of points fixed; but if n be odd, and a point is kept fixed, then a line is thereby also fixed.

Before leaving this subject, I should like to remark that the use of coordinates such as (x, y) above (*circular coordinates*) is often advantageous in geometrical problems that involve metrical relations; thus, properties of the epicycloids, etc., have been very simply obtained from these coordinates by Prof. F. Morley (*Am. Journal of Maths.*, Vols. XIII., XVI.): see also Vol. I., No. 2, of the *Transactions of the American Mathematical Society.*

T. J. I'A. BROMWICH.

93. [K.] Prof. Hill's Eucl. v. and vi., Definition of Ratios, and Incommensurables.

I am glad to have the opportunity of showing that my proposal of *measure* as the definition of ratio is free from the difficulties suggested by Prof. Hill* (Review of Prof. Hill's Eucl. v. and vi. Vol. I. No. 24, p. 411.

ART. 1. If A and B are commensurable so that $A=sC$, $B=rC$, then the measure, $A|B$ say, is the rational fraction s/r, since $A=s/r\,.B$.

When A and B are incommensurable we find, as in the decimal system of weighing, a terminating decimal $\mu_n=\omega\,.a_1a_2\ldots a_n$, such that, the decimal μ'_n being the same as μ_n with a_n+1 for a_n, and μ the true measure $A|B$,

$$A=\mu B>\mu_n B<\mu'_n B.$$

The process is supposed to be carried far enough to show any required difference between μ and any different decimal ν.

ART. 2. *Theorem I.* If the measures $A|B$, $C|D$ are μ, ν, and if, r and s being integers, whenever $rA>sB$, $rC>sD$, and when $<$, $<$, then $A|B=C|D$.

*The less important points raised by Prof. Hill are passed over for lack of space, as well as a detailed account of measurement in Art. 1 justifying (*a*) in Art. 2.

Suppose $\mu > \nu$. Find (Prof. Hill's Prop vii.) integers r, s such that

$$r\mu > s.1 > r\nu \quad \ldots\ldots (a)$$

$\therefore$ $r\mu B > sB$, but $r\nu D < sD$, $\ldots\ldots (b)$

$\therefore$ $rA > sB$, but $rC < sD$, contrary to hypothesis;

$\therefore$ $\mu \not> \nu$, similarly $\mu \not< \nu$; $\therefore$ $\mu = \nu$.

Note i. Professor Hill uses the same argument (a) in his *Art. Phil. Trans.* Prop. 17; and $r\mu$ is intelligible as an operator in (b) just so far as it is intelligible as a number $> s$ in (a). Prop. vii is a direct result of Archimedes' Principle.

Note ii. Professor Hill's 3 sets of conditions $>$, $=$, and $<$, of Prop. viii. are reducible by Prop. vii. to the 2, $>$ and $<$, of this article.

ART. 3. *Theorem II.* If $A|B = \mu$, the scale $[A, B] = [\mu, 1]$.

For, r and s being integers,

whenever $rA > sB$, $r\mu B > sB$

$\therefore$ $r\mu > s.1$.

Similarly, whenever $rA < sB$, $r\mu < s.1$;

$\therefore$ $[A, B] = [\mu, 1]$

Now when A, B are commensurable $= aN$, bN say, Prof. Hill proves, Prop. xi.,

$$[A, B] = [a, b] = [a/b, 1];$$

and says (Art. 44), "a/b is taken as the measure of $a : b$" *i.e.* $A : B$.

Similarly, we say, "μ is taken as the measure of $\mu : 1$" (or $A : B$).

ART. 4. *Theorem III.* Hence, the measure $A|B$ is also the measure of the ratio $A : B$.

Among the advantages of defining ratio by this measure $A|B$ are:

(i.) We can prove the test (Art. 2), and it is therefore scientifically wrong to assume it; it is much easier to apply, and greater and less ratios become intelligible.

ii. The definition is that assumed in Trigonometry, etc.

iii. A fourth Proportional can be at once constructed.

ART. 5. As to the whole question of incommensurables:—I must point out that (i.) approximations being barred, Euclid's Book V. applies only to those kinds of magnitude for which a 4th Proportional can be found without approximation; these are, rational numbers, and certain figures in Euclidean space; and (ii.) it is significant that in general* when the magnitudes A, B, only are given, the scale $[A, B]$ and Euclid's ratio $A : B$ are indeterminate; since, however, many terms of the multiple order $1.B$, $1.A$, $2B$, etc. have been found, the number of possible continuations, and therefore of possible scales or ratios, is still infinite.†

Arts. 1, 2 above, however, show that the difficulties are largely imaginary, and are in any case contained in Archimedes' Principle, assumed in Euclid's postulates, in V., Def. 5, 7, and in Prof. Hill's Prop. vii. Euclid's and Prof. Hill's systems are therefore equally guilty with others. The merit of Euclid's test lies in applying the method of limits once for all.

I ought to say that I am deeply indebted to the suggestiveness of Prof. Hill's Book, and in particular to Prop. vii., for much of the above note and indeed for the suggestion of the 'measure' definition.

E. BUDDEN.

*The exception rational numbers.

†For instance, to the 3rd term given above, A is only known to be some magnitude between $1.B$ and $2.B$; $\therefore$ the number of possible continuations is infinite, and so on.

PROBLEMS.

[Much time and trouble will be saved the Editor if (even tentative) solutions are sent with problems by their Proposers.]

400. [K. 2. d; 5. d.] A variable straight line meets two fixed straight lines Ox, Oy in A and B, so that the sum or difference of OA, OB is constant: find the envelope of the Euler circle of the triangle OAB, and the locus of its Lemoine point and the three associated points.

Prove also that the circumcircle of OAB passes through another fixed point besides O, and that its Euler line also passes through a fixed point.

E. N. BARISIEN.

401. [I. 3.] Prove that

$$3 \cdot 10^n - 7^{24-n} \equiv 0 \pmod{23}$$

for all values of n. Find a process for obtaining similar modular equations $[a \cdot B^n - c \cdot D^{i-n} \equiv 0 \pmod{P}]$ for other primes. Write the equation in its simplest form and find values for the letters when $P = 97$.

R. W. D. CHRISTIE.

402. [B. 1. a.] Prove that the determinant

$$\begin{vmatrix} a_0, & a_1x, & a_2x^2, & \dots & a_nx^n \\ a_nx^n, & a_0, & a_1x, & \dots & a_{n-1}x^{n-1} \\ a_{n-1}x^{n-1}, & a_nx^n, & a_0, & \dots & a_{n-2}x^{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ a_1x, & a_2x^2, & a_3x^3, & \dots & a_0 \end{vmatrix}$$

is a polynomial of order n in x^{n+1}. F. S. MACAULAY.

403. [K. 4.] Construct geometrically a right-angled triangle having given the base b and the rectangle contained by the hypotenuse and the perpendicular. ISAAC H. TURRELL.

404. [J. 2. c.] Material sufficient to make a solid sphere of radius unity is divided at random into two parts, each of which is made into a solid sphere: show that the expectation of the sum of the radii is $\frac{3}{2}$.

If the material had been divided into three parts, the expectation would be $\frac{27}{14}$; if into four parts, it would be $\frac{81}{35}$, and so on. W. A. WHITWORTH.

405. [K. 12. b. a.] Draw three circles mutually tangent, which shall also touch a given straight line at given points. C. E. YOUNGMAN.

406. [K. 3. a.] OAB is an isosceles triangle, base AB, and PAB is any other triangle on the same base and in the same plane: prove that

$$4 \cdot OA^2 \cdot PA \cdot PB \cdot \cos\tfrac{1}{2}(AOB - APB) + AB^2 \cdot OP^2 = OA^2(PA + PB)^2. \quad \text{(C.)}$$

407. [L¹. 4. a. a.] S, H are the foci of an ellipse, T an external point; SM and HN perpendicular respectively to ST and HT meet the adjacent tangents from T to the ellipse in M and N; SN and HM intersect in L: shew that the perpendicular from T on SH passes through L. (C.)

408. [L¹. 14. a.] If the intersections of opposite sides of a circum quadrilateral of an ellipse lie on a coaxal and similar ellipse, shew that one pair of opposite vertices lie on another coaxal and similar ellipse. (C.)

409. [M¹. 3. j.] Lines are drawn from every point of a curve making an angle β with the tangent at the point: if $(\bar{x}, \bar{y})$ be the centre of curvature at a point (x, y), and (ξ, η) the point corresponding to (x, y) on the envelope of the above lines, find ξ and η in terms of x, y, $\bar{x}$, $\bar{y}$, and β. (C.)

SOLUTIONS.

UNSOLVED QUESTIONS.—171, 275, 279, 283, 285, 326, 336-8, 341, 349, 356, 369, 370, 373, 376-82, 387, 389-99.

Solutions to these, or other questions to which no solution has yet been published, and to 400-9 should be sent as early as possible.

The question need not be re-written; the number should precede the solution. Figures should be **very carefully drawn to a small scale** on a separate sheet.

175. [P. 2. a.] *When a conic S is reciprocated with respect to any point O into a conic S', prove that the reciprocals of the four foci of S are the two pairs of common chords of S' and a point-circle at O.* R. P. ROYSTON.

Solution.

If tangents be drawn to a conic from two points F_1, F_2 to a conic S, and the whole be reciprocated with respect to a point O, then corresponding to F_1, F_2 we get two lines L_1, L_2, and to the other intersections of the tangents to S from F_1, F_2 there correspond the other two pairs of straight lines through the intersections of S with L_1, L_2. In the particular case in which F_1 and F_2 are the circular points at infinity L_1 and L_2 together constitute the point-circle at O so that we obtain the required result.

252. [I. 19. a.] *Every whole number greater than 2 is the shortest side of an infinite number of rational plane triangles, whose sides are integers prime to each other.* ARTEMAS MARTIN.

Solution.

We have to shew that given any integer $a(>2)$ there is an infinite number of pairs of integers $b, c(>a)$ which make $[(b+c)^2-a^2][a^2-(b-c)^2]$ a perfect square, a, b, c having no common factor.

1. a odd. Solutions will be given by putting

$$b-c=1,\ a^2-1=md^2,$$

m (obviously not a perfect square) being taken even. Then it is known that the equation

$$x^2-my^2=1$$

admits of an infinite number of solutions in integers, x being in each case odd. The solution is therefore completed by taking

$$b+c=ax.\quad (x>1).$$

2. a twice an odd integer. In this case we take

$$b-c=4,\quad a^2-16=md^2,$$

so that m is necessarily of one of the forms $4n$, $4n+1$.

$$b+c=ax,$$

where x is obtained by solving in integers the equation

$$x^2-my^2=1,$$

which, as before, admits of an infinite number of solutions in each of which x is odd. So that b and c are both odd.

3. a twice an even integer

$$b-c=2,\quad a^2-4=md^2,$$

$b+c=ax$, where x satisfies

$$x^2-my^2=1.$$

271. [J. 1. a.] *Find the number of ways of dividing a number n into not more than three parts.* C. F. W. SANDBERG (O).

Solution.

We want to find the number of ways in which we can assign positive integral values (including zero) to r, s, t $(r \geqslant s \geqslant t)$, so that

$$r+s+t=n.$$

When only two parts are required the number of ways is obviously $1+\frac{n}{2}$ or $1+\frac{n-1}{2}$ according as n is even or odd, *i.e.*, it is $=I\left(\frac{n+2}{2}\right)$. Thus when in the case of three parts r has an assigned value $\left[\geqslant I\left(\frac{n}{3}\right)\right]$, the number of ways of choosing s, t is $I\left(\frac{n-3r+2}{2}\right)$. Therefore the total number of ways of choosing r, s, t is

$$\sum_{r=0}^{I\left(\frac{n}{3}\right)} I\left(\frac{n-3r+2}{2}\right)$$

$$\left.\begin{aligned} &= \sum_{r=0}^{I\left(\frac{n}{3}\right)}\left(\frac{n-3r+2}{2}\right) - \tfrac{1}{2}I\left(\frac{1+I\left(\frac{n}{3}\right)}{2}\right) \text{ if } n \text{ is even,} \\ &= \sum_{r=0}^{I\left(\frac{n}{3}\right)}\left(\frac{n-3r+2}{2}\right) - \tfrac{1}{2}\left[1+I\left(\tfrac{1}{2}I\left(\frac{n}{3}\right)\right)\right] \text{ if } n \text{ is odd,} \end{aligned}\right\}$$

$$= \tfrac{1}{4}\left[1+I\left(\frac{n}{3}\right)\right]\left[2n+4-3I\left(\frac{n}{3}\right)\right] - \tfrac{1}{2}\left[I\left(\frac{n+1}{2}\right) - I\left(\frac{n+1}{3}\right)\right]$$

whether n be even or odd,

$$= \tfrac{1}{12}[(n+3-s)(n+4+s) - \overline{n+1} + 3t' - 2s']$$

where s, s', t' are the remainders obtained on dividing $n, n+1, n+1$ by 3, 3, 2 respectively,

$$= \tfrac{1}{12}[(n^2+6n+11) - s - s^2 - 2s' + 3t'] = 1 + I\left(\frac{n(n+6)}{12}\right)$$

$$\because \quad -6 \geqslant 3t' - s - s^2 - 2s' \geqslant 1.$$

287. *All the quantities being real, prove that the limits of*

$$\left[\sum_{r=1}^{p-1} x^2_{qr} - \sum_{r=1}^{p-2} x_{qr}x_{qr+q}\right] \Big/ \left[\sum_{r=1}^{pq-1} x_r{}^2 - \sum_{r=1}^{pq-2} x_r x_{r+1}\right]$$

are o and q. F. S. MACAULAY

Solution by W. E. HARTLEY.

We have $$\sum_{m=1}^{n} y^2{}_m - \sum_{m=1}^{n-1} y_m y_{m+1} = \tfrac{1}{2}\left[y_1{}^2 + y^2{}_m + \sum_{m=1}^{n-1}(y_m - y_{m+1})\right]$$

and $$\left[\sum_{m=1}^{n} y_m\right]^2 = n\sum_{m=1}^{n} y^2{}_m - \sum_{s=1}^{n-1}\sum_{t=s+1}^{n}(y_s - y_t)^2.$$

The quantity under consideration is therefore equal to

$$\frac{x_q{}^2 + x^2{}_{q(p-1)} + \sum_{r=1}^{p-2}(x_{qr} - x_{qr+q})^2}{x_1{}^2 + x^2{}_{pq-1} + \sum_{r=1}^{pq-2}(x_r - x_{r+1})^2},$$

and is therefore necessarily positive, [and by making $x_{qr}=0 (r=1 \dots \overline{p-1})$ (while the other x's are finite), takes value 0,]

$$=q\Big[x_1^2+\sum_{r=1}^{q-1}(x_{r+1}-x_r)^2\Big]+q\Big[\sum_{r=q(p-1)}^{pq-2}(x_r-x_{r+1})^2+x_{pq-1}^2\Big]+q\sum_{r=q}^{qp-q-1}(x_r-x_{r+1})^2$$

$$-\text{sum of squares}\Big/\Big[x_1^2+x_{pq-1}^2+\sum_{r=1}^{pq-2}(x_r-x_{r+1})^2\Big]$$

$$=q\Big[x_1^2+x_{pq-1}^2+\sum_{r=1}^{pq-2}(x_r-x_{r+1})^2\Big]-\text{sum of squares}\Big/\Big[x_1^2+x_{pq-1}^2+\sum_{r=1}^{pq-2}(x_r-x_{r+1})^2\Big]$$

so that the upper limit is q. Hence the limits of the quantity under consideration are o and q.

306. [J. 1. a.] *A, B, and C have n articles to divide between them subject to the condition that B must not have more articles than either A or C. Show that the number of arrangements is given by the coefficient of x^n in the expansion of $(1-x)^{-2}(1-x^2)^{-1}$. If, however, the condition be that B is not to have fewer than either A or C, show that the generating function is*

$$(1+x^2)(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}.$$

P. A. MACMAHON.

Solution.

We want to find the number of ways in which we can choose positive integers α, β, γ so that $\alpha+\beta+\gamma=n$, $\beta \not> \alpha$, $\beta \not> \gamma$. We may therefore write $\alpha=\beta+a$, $\gamma=\beta+c$, and we have now to find the number of ways of choosing positive integers (including zero) a, β, c such that $a+3\beta+c=n$. The required number is obviously the coefficient of x^n in the expansion of

$$(1+x+x^2+\dots)(1+x^3+x^6+\dots)(1+x+x^2+\dots),$$

i.e. of $$(1-x)^{-2}(1-x^3)^{-1}.$$

In the second case let a, $a+b$, $a+b+c$ be the three parts, a, b, c being all positive integers (including zero), and $(a+b+c)$ being B's share. We have then to determine the number of ways of assigning positive integral values (zero included) to a, b, c so that $3a+2b+c=n$, each arrangement in which b is not zero being counted twice, as either A or C may have $a+b$. The number is the coefficient in the expansion of

$$(1+x+x^3+\dots)(1+3x^2+3x^4+\dots)(1+x+x^2+\dots)=\frac{1+x^2}{(1-x)(1-x^2)(1-x^3)}.$$

307. [I. 2. b.] *The shortest side of a rational plane triangle having integral sides prime to each other cannot be less than 3.*

ARTEMAS MARTIN.

Solution.

If a triangle have integral sides the shortest of which is 1, then the other two must be equal. Let each be n: then the area of the triangle is

$$\tfrac{1}{2}\sqrt{n^2-(\tfrac{1}{2})^2}=\tfrac{1}{4}\sqrt{4n^2-1},$$

which can never be rational since every odd square is of the form $4p+1$.

If the shortest side be 2, then either

(i) the other two are equal; let each be n: then the area is

$$1 \cdot \sqrt{n^2-1},$$

which can never be rational.

Or (ii) the other two differ by 1; let them be n, $n+1$: then the area is

$$[\tfrac{1}{4}\sqrt{\{(b+c)^2-a^2\}\{a^2-(b-c)^2\}}=]$$
$$\tfrac{1}{4}\sqrt{\{(n+2)^2-(n+1)^2\}\{(n+1)^2-(n-2)^2\}}$$
$$=\tfrac{1}{4}\sqrt{3(2n+3)(2n-1)}$$
$$=\tfrac{1}{4}\sqrt{4(3n^2+3n-2)-1},$$

which can never be rational.

Thus the shortest side of a rational plane triangle having integral sides cannot be less than 3.

311. [I. 1.] *The number abcd is such that* $(ab+cd)^2=abcd$. *What four-figure numbers possess this property?* R. PENDLEBURY.

Solution by W. E. HARTLEY, C. E. YOUNGMAN.

We have $(ab+cd)^2=abcd$,

$\therefore (ab+cd)(ab+cd-1)=ab00-ab$
$=ab\times 99.$

$\therefore$ if n denote $ab+cd$

$n(n-1)$ is divisible by 99;

also in order that n^2 may be a four-figure number n must lie between 32 and 99—both limits included. Taking the numbers of the form $11p$ or $11p+1$ in these limits, the only ones which make $n(n-1)$ divisible by 9 as well as 11 are 45, 55 and 99: and these give the numbers

2025, 3025, 9801 (to which we might add 0001).

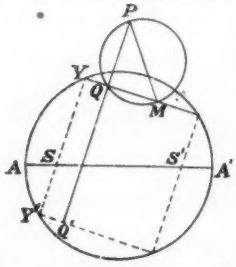
[We see the reasoning above is reversible, so that each of the numbers n which makes $n(n-1)$ divisible by 99 gives one of the four-figure numbers we are seeking, viz. $abcd$ where $n(n-1)=99\times ab$, and $cd=n-ab$.]

320. [L¹. 16. b.] (*a*) *A fixed point P is joined to a point M on a conic, and a circle is described on PM as diameter. Find the envelope of the circles as M moves on the conic, and discuss the cases in which P is* (1°) *at the centre;* (2°) *a focus;* (3°) *a vertex.*

[L¹. 4. c.] (*b*) *Given two conics C, C′, a tangent to C cuts C′ in A and B. Find the locus of the intersections of the tangents to C from A and B.* E. N. BARISIEN.

Solution by C. E. YOUNGMAN.

(*a*) Draw the tangent at M to the conic C, and on it project P into Q. Then Q lies on the circle PM, and also on the consecutive circle PM' for M' lies on MQ; thus the envelope is the locus of Q, or the pedal of C for P. A convenient construction for points QQ' on the curve is:—Take S a focus of C, and $AA'YY'$ the auxiliary circle; draw any chord YY' through S, and a parallel through P, on which project YY' into QQ'. From this follow various properties: the curve has a double point at P, where the tangents are perpendicular to those from P to C; it has (for central conics) no real points at infinity; it cuts the circle AA' on PS and PS'; the tangents to it from P are parallel to those from S to the circle AA', and perpendicular to the asymptotes of C; and it touches C at the four points whose normals pass through P. There cannot be five collinear points Q, for then the five lines QM would touch a parabola with focus P, as well as a conic with



focus S. Hence the curve is a quartic, or, when C is a parabola, a cubic. Inverted from P, it becomes a conic—the polar reciprocal of C for the circle of inversion. Hence if C is a circle the curve Q is a limaçon.

In case 1° the curve is a lemniscate; in 2° the auxiliary circle; in 3° the inverse of a parabola—a cardioid when C is a circle, a cissoid when C is a parabola. See Williamson's *Differential Calculus*, chaps. xiv. and xv., Exx.

(*b*) The locus is a conic touching the common tangents of C and C'. For from Poncelet's Theorem, by first reciprocating the circles into confocal conics, and then projecting these into conics touching four fixed lines, we get: "If a variable polygon circumscribes a conic C, while all its corners but one trace conics of the same four-tangent system with C, then that one also traces another conic of the system." And the case in question is that of a triangle.

327. [B. 2. a.] *Show that there exist linear functions u, v of x, y, z which will render*

$$u\begin{vmatrix} x^2, & y^2, & z^2 \\ xx', & yy', & zz' \\ \alpha^2, & \beta^2, & \gamma^2 \end{vmatrix} + v\begin{vmatrix} x^2, & y^2, & z^2 \\ xx'', & yy'', & zz'' \\ \alpha^2, & \beta^2, & \gamma^2 \end{vmatrix} \text{ divisible by } \begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix};$$

and determine u, v. W. J. Johnston.

Solution.

Denoting the three determinants by Δ_1, Δ_2, Δ respectively, we see that Δ and Δ_1 vanish when for x, y, z we write x', y', z', while—α^2, x'', x'', etc. being general—Δ_2 does not vanish. Hence to satisfy the conditions we must have

$$v=\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ \lambda, & \mu, & \nu \end{vmatrix}, \quad \text{and similarly } u=\begin{vmatrix} x, & y, & z \\ l, & m, & n \\ x'', & y'', & z'' \end{vmatrix}.$$

But Δ vanishes if $x=x'+\rho x''$, etc. whatever ρ may be; these values of x, y, z give

$$u\Delta_1+v\Delta_2=\rho\begin{vmatrix} xx', & yy', & zz' \\ xx'', & yy'', & zz'' \\ \alpha^2, & \beta^2, & \gamma^2 \end{vmatrix}\times\begin{vmatrix} l-\lambda, & m-\mu, & n-\nu \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix};$$

and this vanishes for all values of ρ if, and only if, $\lambda-l=ax'+bx''$, etc. Hence,

$$u=\begin{vmatrix} x, & y, & z \\ l, & m, & n \\ x'', & y'', & z'' \end{vmatrix}; \quad v=\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ l+bx'', & m+by'', & n+bz'' \end{vmatrix},$$

l, m, n being quite arbitrary. But we notice that the value of u is unaltered by writing for $l, m, n, l+bx''$, etc. so that we may say u, v are always $[x, m, z']$, $[x, y' n]$, but corresponding to a given value of u, v is indefinite in so far that we may add to it any multiple of Δ.

344. [L¹. 10. b.] *OP, OQ, OR are normals to a parabola, focus S, vertex A. Prove that $\Sigma A\hat{S}P=2n\pi+2A\hat{S}O$.* R. W. Genese.

Solution.

The line $y=mx-2am-am^3$

is normal to the parabola $y^2=4ax$,

and if P be the point at which it is normal $\angle ASP=-2\tan^{-1}m$.

The values of m which make the normal go through a fixed point O are the roots of the cubic.

$$am^3+m(2a-x)+y=0.$$

$$\therefore\ \Sigma m=0,\quad \Sigma m_2m_3=2-\frac{x}{a},\quad m_1m_2m_3=-\frac{y}{a};$$

m_1, m_2, m_3 being the three roots.

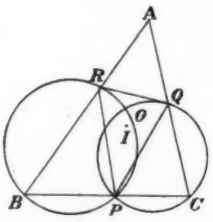
$$\therefore\ \tan\angle ASO=\frac{y}{a-x}=\frac{-m_1m_2m_3}{\Sigma m_2m_3-1}=-\frac{\Sigma m-m_1m_2m_3}{1-\Sigma m_2m_3}$$
$$=-\tan(\Sigma\tan^{-1}m)$$
$$=\tan(\tfrac{1}{2}\Sigma\hat{ASP})$$
$$\therefore\ \Sigma\hat{ASP}=2n\pi+2\hat{ASO}.$$

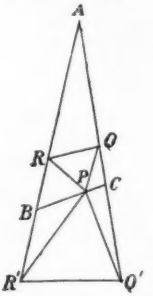
345. [L¹. 2. c.] *Find geometrically the locus of the in-centres of triangles, vertices on three given lines, and equiangular with a given triangle.* J. F. HUDSON.

Solution.

Let ABC be the triangle formed by the three lines; $A'B'C'$ the triangle to which PQR is to be similar; O the point at which BC, CA, AB subtend angles respectively equal to $A+A'$, $B+B'$, $C+C'$. P any point in BC; Q the intersection of CA with $\odot CPO$; R the intersection of BA with $\odot BPO$.

Then $\angle RPQ=\angle RPO+\angle OPQ=\angle RBO+\angle OCQ$
$=\angle BOC-\angle BAC=A'$.





Also POR, POQ being supplementary to B, C, QOR is supplementary to A; and angles Q, R are respectively equal to B' and C', so that PQR is a triangle satisfying the required conditions (conversely if PQR satisfy conditions, $\odot^s AQR$, BRP, CPQ cointersect at O). Since PO, QO, RO make fixed angles with the sides PQR, we have, if $P_0Q_0R_0$ be the pedal triangle of O, I_0 its incentre, I the incentre of PQR,

$$OI_0 : OI=OP_0 : OP,\text{ and }\angle IOI_0=\angle POP_0;$$
$$\therefore\ \angle OI_0I=OP_0P$$

a right angle, so that the locus of I is a line through I_0 perpendicular to OI_0.

If the triangle be turned over so as to be perversely similar to $A'B'C'$, we get a new locus corresponding to the angles $-A'$, $-B'$, $-C'$ instead of $+A'$, $+B'$, $+C'$.

347. [P. 3. b.] *A variable circle, which inverts from each of two fixed points into circles of constant radii, envelopes a pair of circles.* C. E. M'VICKER.

Solution

Let A, B be the fixed centres of inversion. Then since

radius of any circle C : radius of the inverse circle C' with respect to A
= tangent to C from A : tangent to C' from A
= sq. of tangent to C from A : sq. of radius of inversion,

and similarly for the other inversion; we have that the tangents to C from A and B are in a constant ratio. Let L be the limiting point of the system of coaxal circles determined by circle C and line AB. Then $AL : BL$ is a fixed ratio. Hence L describes a circle with centre on AB, and L' the inverse of L with respect to A also describes a circle with its centre O on AB. Now a system of coaxal circles inverts with respect to any point on the radical axis into another system with the same radical axis, the limiting points inverting into the limiting points of the new system. Hence the tangent from O to $C' = OL'$ a constant, and the radius of C' is constant. Hence C' envelopes a pair of circles with a common centre O. Hence C also envelopes a pair of circles.

Solution by C. E. YOUNGMAN.

Let A, B be the fixed points, and V, V' two positions of the variable circle, From either A or B, say A, V and V' invert into equal circles O, O'; $\therefore$ A and B lie on a circle coaxal with V, V' (Casey's *Sequel*, VI. iv. 5); $\therefore$ the inverse of this—a line cutting AB at B' the inverse of B—is coaxal with O, O' (*Ib.* VI. v. 6, Cor. 6); $\therefore$ tangents from B' to O and O' are equal; *i.e.* the tangent from B' to O is constant. Hence O (being of fixed size) touches two fixed circles with centre B', and $\therefore$ V touches other two coaxal with A and B.

350. [K. 2. a.] *The triangle formed by the Simson lines of A', B', C' with respect to a triangle ABC is similar to the triangle DEF, where A', B', C' are the mid points of BC, CA, AB respectively, and D, E, F the points of contact of the inscribed circle of ABC.* W. J. G.

Solution by N. QUINT.

This theorem is a particular case of that given by Mr. J. Alison (*Proc. Edin. M. Soc.*, III., p. 85): "if E, F, G be the vertices of any triangle inscribed to a circle ABC, the Simson lines of E, F, G form a triangle similar to EFG"; for as is easily seen, the triangle $A'B'C'$ is equiangular with the triangle DEF formed by joining the points of contact. It may be remarked that—as in many questions dealing with Simson lines—the theorem still holds if we replace the Simson lines by the *general* Simson lines, obtained by taking the oblique projections of A', B', C'. For it has been shown ("The general Wallace line of an in-polygon" (Quint.) *Nieuw Archief voor Wiskunde*, III. p. 153, 1897) that the angle between the general Simson lines for O, O' on the circle for any inscribed triangle is equal to half the arc OO'.

360. [K. 10. e.] *In a semicircle ABA', centre C, a chord $A'B$ is drawn, and a radius CD parallel to $A'B$. Find the locus of intersection of $CB, A'D$.*

C. BICKERDIKE.

Solution.

Let O be the intersection; $O\hat{C}A = \theta$, $OC = r$, $CA = a$,

then $$CO : CB :: CD : CD + A'B$$

or $$r : a :: a : a + 2a\cos\frac{\theta}{2};$$

$$\therefore\ r\left(1 + 2\cos\frac{\theta}{2}\right) = a,$$

the equation to the locus.

362. [K. 10. e.] *If an arc AB of a circle be divided equally in M and unequally in P, find a relation between the cords PA, PB, PM, AM analogous to Euc.* II. 5. W. J. JOHNSTON.

Solution by PROPOSER.

Let the point P lie between M and B. Take a point Q in the arc AM, such that arc AQ=arc MP. Then arc AM=arc QP, and arc QM=arc PB. Apply Ptolemy's Theorem to the quadrilateral $AQMP$. Then noticing that the chords of the preceding pairs of equal arcs are equal, we obtain the relation

$$AP \cdot PB + MP^2 = MB^2.$$

363. [L¹. 3. a.] *Given two points on an ellipse and the positions of its axes, construct the lengths of the axes.* A. LODGE.

Solution by R. B. WORTHINGTON.

Construction. Let PP' meet axis major CMM' in T. Join PM', $P'M$, meeting in H. Draw $NHR \parallel PM, PM'$, and meeting the semicircle CRT in R, PP' in Q.

Then shall $CR = \frac{1}{2}$ axis major.

For $TPQP$ is harmonic, $\therefore$ NQR is the polar of T, both for ellipse and auxiliary ⊙.

TRC is a right angle; $\therefore$ R is evidently on the auxiliary circle. $\therefore$ CR is the $\frac{1}{2}$ axis major.

Let MP meet auxiliary in p, and take $CB : CR = MP : Mp$; CB is $\frac{1}{2}$ axis major.

365. [K. 20. d.] *An ass is tied to a peg in the centre of a rectangular plot of grass, sides a, $2a$; if he can graze over just half the area, show that the length of his halter must be $\frac{1}{2}a$ cosec 59° 4′ 51·6″ very nearly.* C. E. M'VICKER.

Solution by PROPOSER; R. F. DAVIS; E. M. RADFORD; J. F. HUDSON; and others.

It is easily seen that we are led to the equation

$$2\theta + \sin 2\theta + 2\cos 2\theta = 2.$$

By experiment $\theta = \dfrac{\pi}{3}$ nearly, $= \dfrac{\pi}{3} + x$ (say).

Hence $$2\left(\frac{\pi}{3}+x\right) + \sin\left(\frac{2\pi}{3}+2x\right) + 2\cos\left(\frac{2\pi}{3}+2x\right) = 2.$$

Expanding the sine and cosine we have

$$x = -\frac{18-4\pi-3\sqrt{3}}{6(2\sqrt{3}-1)} + \left(\frac{2-\sqrt{3}}{2\sqrt{3}-1}\right)x^2 + \frac{2(2\sqrt{3}+1)}{3(2\sqrt{3}-1)}x^3 \ldots;$$

or, writing β for the first term:

$$x = -\beta + \cdot1087411x^2 + 1\cdot2077698x^3 \ldots,$$

from which by continued approximation

$$\begin{aligned} x &= -\beta + \cdot1087411\beta^2 - 1\cdot2314190\beta^3 \ldots \\ &= -\cdot0160624 + \cdot0000280 - \cdot0000051 \ldots \end{aligned}$$

[the term in β^4 will be found to be $< \cdot01''$];

$$\begin{aligned} \therefore\ \theta &= 1\cdot0311580 \ldots \text{ radians} \\ &= 59^\circ\ 4'\ 51\cdot6''. \end{aligned}$$

[This problem was set (in a modified form) in "Tit-Bits" as a *puzzle*!]

366. [K. 3. b.] *ABC, DBC are two equilateral triangles on the same base BC. A point P is taken on the circle, centre D, radius DB or DC. Show that PA, PB, PC are the sides of a right-angled triangle.* E. M. RADFORD.

Solution by C. V. DURELL and others.

AD bisects BC at right angles. $OD = AB\cos 30°$; $BO = \frac{AB}{2}$.

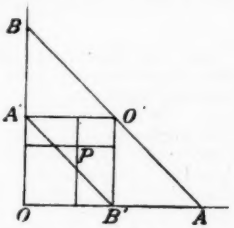
$$\therefore\quad PB^2 + PC^2 = 2PO^2 + 2OB^2 = 2PO^2 + \frac{AB^2}{2},$$

$$PA^2 + PD^2 = 2PO^2 + 2OD^2 = 2PO^2 + \frac{3AB^2}{2}.$$

But $PD^2 = BD^2 = AB^2$; $\therefore$ etc.

367. [J. 2. c.] *A straight line of length unity is divided at random into there parts, subject to the condition that they may form the sides of a triangle. Show that the expectation of the area of the circle inscribed in the triangle is* $\frac{\pi}{240}$.

W. ALLEN WHITWORTH.



Solution by C. E. M'VICKER.

Let a, b, c be the parts of the line.

$$\text{Area of circle} = \frac{\pi}{s}(s-a)(s-b)(s-c)$$

$$= \frac{\pi}{s}(s-a)(s-b)(a+b-s).$$

$$\therefore \text{ expectation of area} \equiv p = \frac{\pi}{s}\iint(s-a)(s-b)(a+b-s)\,da\,db \Big/ \iint da\,db,$$

where $a < s$; $b < s$; $a+b > s$.

To exhibit the limits of integration geometrically, take a representative point P whose coordinates referred to two rectangular axes are a, b; make $OA = OB = 2s$, the length of the line. It will be seen from the preceding inequalities that P must lie on the triangle $O'A'B'$, whose vertices are the mid-points of the sides of OAB.

Taking $O'A'$, $O'B'$ as new axes of reference,

$$p = \frac{\pi}{s}\iint xy(s-x-y)\,dx\,dy \Big/ \frac{s^2}{2} = \frac{2\pi}{s^2}\iint xy(s-x-y)\,dx\,dy,$$

the integration extending over $O'A'B'$.

$$\therefore\quad p = \frac{2\pi}{s^3}\cdot\frac{s^5}{120} = \frac{\pi s^2}{60} = \frac{\pi}{240}. \quad [\because\ s = \tfrac{1}{2}.]$$

372. [K. 13. a.] *A plane figure ABC... receives any rigid-body displacement in space to the position A'B'C'.... Prove that the mid-points of AA', BB', CC',... are co-planar, and that the planes bisecting these lines at right angles have a common point.* C. E. M'VICKER.

Solution by J. BLAIKIE.

(1) Join AA', BB',... and let A'', B''... be the mid-points of the joining lines. Take any point O as origin, and employ the method of *Vectors*. Since $ABCD$ is congruent with $A'B'C'D'$, if we take

$$AD = mAB + nCD, \text{ we have also } A'D' = mA'B' + nC'D' \ldots\ldots\ldots\ldots(1).$$

$$\therefore\quad OD = OA + AD = OA + mAB + nCD.$$

$$OD' = OA' + A'D' = OA' + mA'B' + nC'D'.$$

$$OD'' = \tfrac{1}{2}(OD + OD') = \tfrac{1}{2}(OA + OA') + \frac{m}{2}(AB + A'B') + \frac{n}{2}(CD + C'D').$$

Now $A''B''=OB''-OA''=\frac{1}{2}(OB+OB')-\frac{1}{2}(OA+OA')$
$=\frac{1}{2}(OB-OA)+\frac{1}{2}(OB'-OA')=\frac{1}{2}(AB+A'B')$.
$\therefore\ OD''=OA''+mA''B''+nC''D''$;

from which we see that D'' is co-planar with A'', B'', C''. Similarly for other points. It is to be observed that this proof holds good when the figures are similar but not equal, and in all cases where equation (1) is satisfied.

(2) Since O is any point, we may assume it to be the point of intersection if the normal planes to AA', BB', CC', through A'', B'', C''. Then $OA=OA'$, etc., and the tetrahedron $OABC$ is either congruent with, or is the image of, the tetrahedron $O'A'B'C'$. In the latter case we may take a point $\bar{O}$, the image of O in the plane $A'B'C'$, and the tetrahedrons $OABC$, $O'A'B'C'$ are congruent. Hence, by superposition, either $OD=OD'$, or $OD=O'D'=OD'$. $\therefore$ OD'' is perpendicular to DD'. Similarly for other points.

385. [K. 13. a.] *Show that any displacement of a rigid body is equivalent to a series of reflexions in plane mirrors of the points of the body.* W. J. JOHNSTON.

Solution by J. BLAIKIE.

Let A, B, C be three points in the body, and let A', B', C' be their positions after displacement. Join AA', and draw a plane through its mid-point perpendicular to it. Let $A'B''C''$ be the image of ABC in this plane. Join $B''B'$. Then since $A'B''=A'B'$, the plane through the mid-point of $B''B'$ at right angles to it will pass through A'. Let the image of $A'B''C''$ in this plane be $A'B'C'''$. Then it is clear that $A'B'C'$ is the image of $A'B'C'''$ in a plane through $A'B'$ bisecting $C'C'''$ at right angles. If D be a fourth point in the rigid body, a fourth reflexion in the plane $A'B'C'$ will be required to make the final image of D agree with its displaced position D'.

386. [L_1. 3. c.] *If a circle through the centre of an ellipse cut pairs of conjugate diameters in A, A'; B, B';... then shall the chords AA', BB', ... all pass through a fixed point.* A. LODGE.

Solution by W. F. BEARD.

Conjugate diameters from a pencil in involution. Thus the pencil $C(AA'BB'\ldots)$ is in involution, and therefore (*vide* Russell's *Pure Geom.*, ch. XX., § 2), AA', BB', etc., are concurrent.

Solution by R. F. DAVIS.

Let CP, CD be conjugate semi-diameters of the ellipse, CD touching the given circle at C. Along the tangent at P take any points Q, Q' such that $QP.PQ=CD^2$; then CQ, CQ' are conjugate diameters intersecting the given circle in A, A'. Produce CP to U so that $CP.PU=CD^2$.

Then the variable circle CQQ' passes through a fixed point U. Hence if the constant of inversion from C be chosen so that the tangent at P inverts into the given circle, AA' passes through a fixed point on CP.

This theorem obviously gives by orthogonal projection Frégier's property.

28. [K. 1. c.] *Find a point P in the base BC of a triangle ABC, such that $AP^2-BP.PC$ is a max. or min. Discuss the different cases.* E. M. LANGLEY.

Solution by G. HEPPEL.

Draw AN perpendicular to BC. Bisect BC in O. Take Q on BC such that $ON=QN\sqrt{2}$. The cases are distinguished as $AQ \gtreqless BO$.

If $OB=OC=a$; $ON=h$; $AN=k$; $OP=x$;
$u=AP^2-BP.PC=k^2+\frac{1}{2}h^2-a^2+\frac{1}{2}(h-2x)^2=AQ^2-BO^2+\frac{1}{2}(h-2x)^2$.

If $u=0$, $x=\frac{1}{2}h\pm\sqrt{\frac{1}{2}(BO^2-AQ^2)}$.

If $AQ>BO$; no max. and a min. for $x=\frac{1}{2}h$.

$AQ=BO$; min. is zero.

$AQ<BO$; min. zero for $x=\frac{1}{2}h\pm\sqrt{\frac{1}{2}(BO^2-AQ^2)}$; max. for $x=\frac{1}{2}h$.

32. [P. 3. b.] *If a system of coaxal circles, real limiting points L, L', be inverted from any point E in their centre-join, with radius of inversion* $\sqrt{EL.EL'}$, *then each circle of the system inverts into another circle of the system, and the circle through E inverts into the radical axis of the system, and is the circle of similitude of each pair of mutually inverse circles.* H. D. ELLIS.

Solution by R. F. DAVIS.

Let AOB be the radical axis bisecting LL' at right angles in O. Describe the circle whose centre is O and radius OL, and through E draw any secant EPQ. Let the tangents TP, TQ meet LL' in U and V respectively.

Since $EP.EQ=EL.EL'=$square of radius of inversion, the point P inverts into the point Q. The circle LPQ is its own inverse; and a circle intersecting orthogonally the circle LPQ at P whose centre lies on LL' inverts into a circle also intersecting orthogonally the circle LPQ at Q whose centre lies on LL'. Hence the circle centre U radius UP inverts into the circle centre V radius VQ: these both belong to the given coaxal system having L, L' as limiting points.

The circle of the system passing through E will invert into the circle of the system having an infinitely large radius, that is, the radical axis AB.

Draw TRM perpendicular to LL' meeting the circle LPQ in R: and join ER which will be the tangent at R. Then since $T\{PMQE\}=-1$, $\{UMVE\}=-1$: but E is obviously the external centre of similitude of the circles centres U, V, hence M is the internal centre of similitude of the same circles, and the circle upon EM as diameter their circle of similitude. But the circle upon EM as diameter is the circle of the system passing through E; for $EM.EO=ER^2=EP.EQ$.

37. [R. 4. a.] *Find from statical considerations the length of the bisector of an angle of a triangle.* E. M. LANGLEY.

Solution by PROPOSER.

The resultant of two equal forces of bc units along AB, AC respectively is

$$2bc\cos\frac{A}{2}.$$

These forces are $b.AB$, $c.AC$ respectively, and therefore are equivalent to $(b+c)AD$ along the bisector AD. Hence

$$AD=2bc\cos\frac{A}{2}\Big/(b+c).$$

39. [A. 1.] *Find a number of two digits equal to the square of the tens' digit together with the square of the sum of its digits.* E. M. LANGLEY.

Solution by PROPOSER.

We have to solve $x^2+(x+y)^2=10x+y$.

The roots of this equation regarded as a quadratic in x are real if $25-y(y+8)$ is positive, *i.e.* for $y=0$, 1, or 2; further, the roots are rational if $y=0$ or 1, but irrational if $y=2$. Hence 50 and 41 are the numbers.

42. [K. 5. a.] *I measure a field by pacing. I then make a sketch of the land, using a scale of paces, which is on the scale of 8 inches to a mile. The regulation pace is 32 inches. After drawing my sketch I find that my pace is only 31 inches. What should be the length of a special scale of yards for the sketch to show 1000 yards in all?* J. H. HOOKER.

Solution by PROPOSER.

Length of scale of 1000 yds. for standard pace $=\frac{8}{5\,2\,8\,0}\times1000\times36''=4\frac{6}{11}''$.

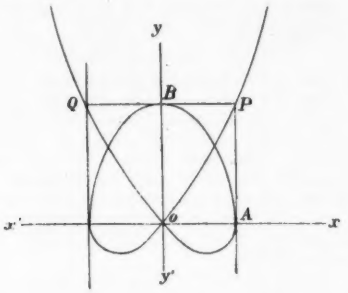
Personal pace $=\frac{31}{32}$ (standard pace);

$\therefore$ a distance in sketch reading x yds. is really $\frac{31}{32}x$ yds.;

$\therefore$ scale must be increased in ratio $\frac{32}{31}$;

$\therefore$ length required $=\frac{50}{11}''\times\frac{32}{31}=4''\cdot69\ldots$.

Read this figure into solution 276 of December number.



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* Will be reviewed shortly.

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